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Irreducible Free Energy Expansion and Overlaps Locking in Mean Field Spin Glasses

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Following the works (9), (11), we introduce a diagrammatic formulation for a cavity field expansion around the critical temperature. This approach allows us to obtain a theory for the overlap's fluctuations and, in particular, the linear part of the Ghirlanda–Guerra relationships (GG) (often called Aizenman–Contucci polynomials (AC)) in a very simple way. We show moreover how these constraints are "superimposed" by the symmetry of the model with respect to the restriction required by thermodynamic stability. Within this framework it is possible to expand the free energy in terms of these irreducible overlaps fluctuations and in a form that simply put in evidence how the complexity of the solution is related to the complexity of the entropy.

KEY WORDS: Cavity field, Ghirlanda–Guerra, stochastic stability, Aizenman-Contucci polynomials

1. INTRODUCTION

The configuration space of the Sherrington–Kirkpatrick model (S–K) is built by N Ising variables $\{\sigma_i\}$ which interact respecting the following Hamiltonian $H(\{\sigma\})^{(1-3)}$:

$$H(\{\sigma\}) = -\frac{1}{\sqrt{N}} \sum_{i<j}^{N,N} J_{ij}\sigma_i\sigma_j - h\sum_i^N \sigma_i \tag{1}$$

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where h is an external scalar field, which, for the rest of the paper, we will set to zero, and J_{ij} are i.i.d. Gaussian variables such that $P[J_{ij}]$:

$$P[J_{ij}] = \prod_{i<j} \left[\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} J_{ij}^2 \right) \right]. \tag{2}$$

Once defined the partition function $Z(\beta, J_{ij})$ as

$$Z_N(\beta, J_{ij}) = \sum_{\{\sigma\}} e^{\frac{\beta}{\sqrt{N}} \sum_{ij} J_{ij}\sigma_i\sigma_j} \tag{3}$$

with the inverse temperature $\beta = T^{-1}$ in proper units, the Gibbs measure $G_N(\sigma; \beta, J_{ij})$

$$G_N(\sigma) = \frac{1}{Z_N} \exp^{-\beta H_N(\sigma)}$$

and all the standard statistical mechanics package, let us introduce also the average over the $\{J_{ij}\}$ denoting by $\mathbf{E}(.)$:

$$\mathbf{E}[f(J_{ij})] = \prod_{i<j} \frac{1}{\sqrt{2\pi}} \int dJ_{ij} f(J_{ij}) \exp\big(- J_{ij}^2/2 \big). \tag{4}$$

We are interested in the explicit expression for the quenched average of the free energy density (we will use the symbol F for the free energy and f for its density):

$$\alpha(\beta) = \lim_{N\to\infty} \alpha_N(\beta) = -\beta \mathbf{f}(\beta) = N^{-1} \mathbf{E} \ln Z_N(\beta, J_{ij}). \tag{5}$$

Let us introduce some basic notation: For a function $f(\{\sigma_i\}_{(i=1,\ldots,N)}, J)$ of the degrees of freedom $\{\sigma\}$ and the noise J we define:

$$\omega[f(\{\sigma\})] = \frac{\sum_{\{\sigma\}} f(\{\sigma\}) e^{-\beta H(\{\sigma\})}}{\sum_{\{\sigma\}} e^{-\beta H(\{\sigma\})}} \text{ as the thermal average,}$$

$\Omega[f(\{\sigma^\alpha\})] = \omega_1[f(\sigma^1)]\omega_2[f(\sigma^2)] \cdots \omega_n[f(\sigma^n)]$ as the generalized thermal average over the replicas of the system,[5]

$\langle f(\{\sigma\}, J) \rangle = E\Omega[f\{\sigma\}]$ as the average first on the thermal weight and than over the coupling,[5]

$q_{\alpha\beta} = N^{-1} \sum_i^N \sigma_i^\alpha \sigma_i^\beta$ as the overlap of two replicas,
$q_{\alpha\beta\gamma} = N^{-1} \sum_i^N \sigma_i^\alpha \sigma_i^\beta \sigma_i^\gamma$ as the overlap over three replicas and so on.

2. CAVITY FIELD: TOWARD A DIAGRAMMATIC FORMULATION

The main idea of the *cavity field* method is to look for an explicit expression of $\alpha(\beta) = -\beta f(\beta)$ upon increasing the size of the system from N particles (the cavity) to $N+1$[3,9,10,13] so that, in the limit of N that goes to infinity

$$[-\beta F_{N+1}(\beta)] - [-\beta F_N(\beta)] = -\beta f(\beta) + O\left(\frac{1}{N}\right)$$

because the existence of the thermodynamic limit[14] implies only vanishing correction of the free energy density.

So, following[11] in a variant nowadays called stochastic stability,[12] let us introduce an extended partition function[10] $Z_{N,t}$ able to manage the interaction with the added spin through a control parameter $t \in [0, \beta^2]$ such that for $t = 0$ we have the classical partition function of N spins while for $t = \beta^2$ we get the expression of the partition function for the larger system with a little temperature shift which vanishes in the thermodynamic limit.

$$Z_{N,t} = \sum_{\sigma_1,\dots,\sigma_N} e^{-\beta H_N(\sigma,J) + \sqrt{\frac{t}{N}} \sum_i^N J_i \sigma_i} \tag{6}$$

when $t = \beta^2$, redefining $J_i \to J_{i,N+1}$ and making the transformation $\sigma_i \to \sigma_i \sigma_{N+1}$ $\forall i$ we obtain the partition function for a system of $N+1$ spin at a scaled temperature β^* such that

$$\beta^* = \beta\sqrt{(N+1)/N} \to \beta \quad \text{for} \quad N \to \infty. \tag{7}$$

Now we will try to formalize some concepts which will be useful developing this version of cavity method:

Proposition 1. *The averages $\langle . \rangle$ are invariant under replica symmetry. In fact if we consider a generic element g of the group **G** of the permutation of s replicas, defining a generic replica (a) and its transformed (a′), the action of g on (a) is such that:*

$$\boldsymbol{G} \ni g : g * (a) = (a') \Rightarrow \langle F(q_{ab})\rangle = \langle F(q_{AB'})\rangle$$

Proposition 2. *The averages $\langle . \rangle$ are invariant under gauge symmetry:*

$$\sigma_i^a \to \epsilon_a \sigma_i^a$$

$$q_{ab} \to \epsilon_a \epsilon_b q_{ab}$$

being $\epsilon_a = \pm 1$

This symmetry is a consequence of the parity of the state Ω and of dichotomy of Ising variables.

Definition 1. We define as filled a polynomial of the overlaps in which every replica appears an even number of times.

Definition 2. We define as fillable a polynomial in which the above property is obtainable by multiplying the polynomial itself for just one overlap of two replicas that is exactly the needed term to fill the expression.

Definition 3. We define as un-fillable a polynomial which is neither filled nor fillable.

Example: the following expressions are filled: q_{12}^2, $q_{12}q_{23}q_{31}$.
Example: the following expression are fillable: q_{12}, $q_{12}q_{23}$.
Example: the following expressions are un-fillable: q_{1234}, $q_{12}q_{23}q_{45}$.

Definition 4. We define the *cavity function* $\Psi(t)$ as:

$$\Psi(t) = \mathbf{E}\big[\ln \omega\big(e^{\frac{\sqrt{t}}{N}\sum_i J_i\sigma_i}\big)\big] = \mathbf{E}\left[\ln \frac{Z_{N,t}}{Z_N}\right] \tag{8}$$

and let F be a generic function of the $\{\sigma\}$ such that,

Definition 5. we define the *generalized Boltzmann state* obtained using the partition function[10] as:

$$\omega_t(F) = \frac{\omega\big(Fe^{\sqrt{\frac{t}{N}}\sum_i^N J_i\sigma_i}\big)}{\omega\big(e^{\sqrt{\frac{t}{N}}\sum_i^N J_i\sigma_i}\big)}. \tag{9}$$

Theorem 1. *In the $N \to \infty$ limit the averages $\langle . \rangle$ of the filled polynomials are t-independent in β average.*

Proof: Without loss of generality we will prove the theorem in the simplest case (for q_{12}^2). Let us write the cavity function as

$$\Psi(t) = \mathbf{E}[\ln Z_{Nt}] - \mathbf{E}[\ln Z_N] \tag{10}$$

and derive it respect to β:

$$\frac{d\Psi(t)}{d\beta} = \frac{\beta N}{2}\big(\langle q_{12}^2\rangle - \langle q_{12}^2\rangle_t\big). \tag{11}$$

We can introduce an auxiliary function $\Upsilon_N(t,\beta)$:

$$\Upsilon_N(t,\beta) = \frac{4}{N} d_{\beta^2}[\Psi_N(t,\beta)]$$

and integrate it in the interval $[\beta_1, \beta_2]$:

$$\int_{\beta_1^2}^{\beta_2^2} \Upsilon_N(t,\beta)d\beta^2 = \frac{4}{N}[\Psi_N(t,\omega(\beta_2)) - \Psi_N(t,\omega(\beta_1))].$$

In the thermodynamic limit $\Psi(t)$ remains limited and the second member goes to zero; so, $\forall[\beta_1^2, \beta_2^2]$ we can always extract a subsequence such that the $\Upsilon_N(t)$ converge to zero in Lebesgue measure.

The next theorem is crucial for this paper, so, for the sake of simplicity we divided it in two part: the first one will be the following lemma and it will make us able to proof the theorem itself which will be showed immediately after.

Lemma 1. *Let $\omega(.)$ and $\omega_t(.)$ be the states defined respectively by the canonical partition function and by the extended one; if we consider the ensemble of index $\{i_1, \dots, i_r\}$ with $r \in [1, N]$, then for $t = \beta^2$ the following relation holds:*

$$\omega_{N,t=\beta^2}(\sigma_{i_1}, \dots, \sigma_{i_r}) = \omega_{N+1}\left(\sigma_{i_1}, \dots, \sigma_{i_r}, \sigma_{N+1}^r\right) + O\left(\frac{1}{N}\right)$$

where r is an exponent, not a replica index, so if r is even $\sigma_{N+1}^r = 1$, while if is odd $\sigma_{N+1}^r = \sigma_{N+1}$.

Proof: Let us write the ω_t for $t = \beta^2$ defining for the sake of simplicity $\sigma = \sigma_{i_1} \cdots \sigma_{i_r}$:

$$\omega_{N,t=\beta^2}(\sigma) = \mathbf{E}\left[\sum_{\{\sigma\}} \frac{1}{Z_{N,\beta^2}} e^{\frac{\beta}{\sqrt{N}} \sum_{i<j} J_{ij}\sigma_i\sigma_j + \frac{\beta}{\sqrt{N}} \sum_i J_i \sigma_i} \sigma\right]. \tag{12}$$

Introducing a sum over σ_{N+1} at the numerator and at the denominator, (which is the same as multiply and divide for 2^N because there is no dependence to σ_{N+1}) and making the transformation $\sigma_i \to \sigma_i \sigma_{N+1}$, the variable σ_{N+1} appears at the numerator and it is possible to build the status at $N+1$ particles. Expanding β^* for large N we have that:

$$\omega_{N,t=\beta^2}(\sigma) = \omega_{N+1}\left(\sigma\sigma_{N+1}^r\right) + O\left(\frac{1}{N}\right). \tag{13}$$

Using this lemma we are able to proof the following:

Theorem 2. *Let Q_{ab} be a fillable polynomial of the overlaps, (this means that $\langle q_{ab} Q_{ab}\rangle$ is filled). We have:*

$$\lim_{N\to\infty} \lim_{t\to\beta^2} \langle Q_{ab}\rangle_t = \langle q_{ab} Q_{ab}\rangle$$

Proof: We write the average splitting the dependence from the non filled replicas a,b to the others:

$$Q_{ab} = \sum_{ij} \frac{\sigma_i^a \sigma_j^b}{N^2} Q_{ij}(\sigma). \tag{14}$$

We have indicated with $Q_{ij}(\sigma)$ the product of the non filled replicas. Factorizing the state Ω we obtain:

$$\langle Q_{ab}\rangle_t = \frac{1}{N^2}\mathbf{E}\Big[\sum_{ij}\Omega_t\big(\sigma_i^a\sigma_j^b Q_{ij}(\sigma)\big)\Big] \tag{15}$$

$$= \frac{1}{N^2}\mathbf{E}\Big[\sum_{ij}\omega_t\big(\sigma_i^a\big)\omega_t\big(\sigma_j^b\big)\Omega_t(Q_{ij})\Big]. \tag{16}$$

Now we write the last expression for $t = \beta^2$; using the lemma, the states acting on the replicas a and b are

$$\omega_{t=\beta^2}\big(\sigma_i^a\big) = \omega\big(\sigma_i^a\sigma_{N+1}^a\big) + O\left(\frac{1}{N}\right) \tag{17}$$

while the remaining product state Ω_t continue to work on a even number of replicas and is not modified

$$\Omega_{t=\beta^2}(Q_{ij}) = \Omega(Q_{ij}). \tag{18}$$

Putting all the replicas in a unique product state we have:

$$\omega\big(\sigma_i^a\sigma_{N+1}^a\big)\omega\big(\sigma_i^b\sigma_{N+1}^b\big)\Omega(Q_{ij}) = \Omega\big(\sigma_i^a\sigma_j^b\sigma_{N+1}^a\sigma_{N+1}^b Q_{ij}\big). \tag{19}$$

Using replica symmetry we can write the index $N+1$ as a "k" dumb one; pass on, we can sum on all "k" from 1 to N and divide for N because the terms with k equal to one of the index i, j are of order $O(\frac{1}{N})$ and became irrelevant in the $N \to \infty$ limit. So:

$$\langle Q_{ab}\rangle_{\beta^2} = N^{-3}\mathbf{E}\left[\sum_{ijk}\Omega\big(\sigma_k^a\sigma_k^b\sigma_i^a Q_{ij}\sigma_j^b\big)\right] + O\left(\frac{1}{N}\right) \tag{20}$$

and in the thermodynamic limit we have the proof.

Let us remember (this will be useful soon) an important theorem, due to F. Guerra, stating that the streaming equations for the cavity fields are[(6)]:

Theorem 3. *Let $F_s \in \{A_s\}$, where $\{A_s\}$ is the algebra built by the overlaps of s replicas; the following streaming equation for F holds:*

$$\partial_t\langle F_s\rangle_t = \left\langle F_s\left(\sum_{a,b} q_{a,b} - s\sum_{a=1}^{s} q_{a,s+1} + \frac{s(s+1)}{2} q_{s+1,s+2}\right)\right\rangle_t. \tag{21}$$

By now we define abstract graphs in this way: We introduce numbered vertexes to label replicas of the system and lines between them to label their overlaps; for example $\langle q_{12}\rangle_t$ will be associated to the graph $1 \longleftrightarrow 2$, $\langle q_{12}^2\rangle_t$ to $1 \bigcirc 2$ and so on. It is obvious that there is a one to one connection between polynomials and graphs.

3. FREE ENERGY AND CAVITY FUNCTION

Now we want to show how it is possible to expand the SK free energy via these graphs. Look carefully at the energy of the system, using thermodynamic relations we have:

$$\mathbf{E}[\omega(H_N)] = -N\partial_\beta \alpha_N(\beta) = -\mathbf{E}\left[\frac{\partial_\beta Z_N(\beta, J)}{Z_N(\beta, J)}\right] = \frac{-N\beta}{2}\big(1 - \langle q_{12}^2\rangle\big).$$

So in our vocabulary:

Proposition 3. *Internal energy of SK model is:*

$$\mathbf{E}[u_N(\beta)] = \frac{1}{N}\mathbf{E}[\omega_J(H_N(\sigma, J))] = -\frac{\beta}{2}(1 - \langle \bigcirc \rangle).$$

Theorem 4. *Assuming that* $\lim_{N\to\infty} \Psi_N(\beta) = \Psi(\beta)$ *exists uniformly in every compact* $0 \le \beta \le \beta^*$ *than the following relation holds:*

$$\alpha(\beta) + \frac{\beta}{2}\alpha'_\beta = \ln 2 + \Psi(\beta) \tag{22}$$

where $\Psi_N(\beta)$ *is the cavity function previously defined.*

This theorem give us a connection between the cavity function in the limit $t \to \beta^2$, the free energy and the internal energy of the system.

Proof: Consider the partition function of a system of $(N+1)$ spins and point out with β the effective temperature and with β^* the scaled one:

$$Z_{N+1} = \sum_{\{\sigma_{1,\dots,N+1}\}} e^{\frac{\beta}{\sqrt{N+1}}\Sigma_{i<j}^{N+1} J_{ij}\sigma_i\sigma_j} = 2\sum_{\{\sigma_N\}} e^{\frac{\beta^*}{\sqrt{N}}\Sigma_{i<j}^{N} J_{ij}\sigma_i\sigma_j} e^{\frac{\beta}{\sqrt{N+1}}\sum_i^N J_i\sigma_i}. \tag{23}$$

Multiplying and dividing for $Z_N(\beta^*)$, taking the logarithm, subtracting from every members the quantity $\ln Z_{N+1}(\beta^*)$ and expanding $\ln Z_{N+1}(\beta)$ around $\beta = \beta^*$ we have:

$$\ln Z_{N+1}(\beta) - \ln Z_{N+1}(\beta^*) = (\beta - \beta^*)\partial_{\beta^*} \ln Z_{N+1}(\beta^*) + O(d\beta^2)$$

being:

$$(\beta - \beta^*) = \beta^* \left(\sqrt{\frac{N+1}{N}} - 1 \right) = \frac{\beta^*}{2N} + O(N^{-1}).$$

Substituting β with β^* inside the state ω apart corrections $O(N^{-1})$ we have:

$$\ln Z_{N+1}(\beta^*) + (\beta - \beta^*)\partial_{\beta^*} \ln Z_{N+1}(\beta^*) = \ln 2 + \ln Z_N(\beta^*)$$
$$+ \ln \omega_N^{(\beta^*)}\big(e^{\frac{\beta}{\sqrt{N+1}} \sum_i J_i \sigma_i}\big) + O(N^{-1}).$$

Taking the average $\mathbf{E}$, using the variable α and renaming $\beta^* \to \beta$ in the thermodynamic limit we get:

$$\alpha(\beta) + \frac{\beta}{2} \partial_\beta \alpha(\beta) = \ln 2 + \Psi(\beta). \tag{24}$$

and this is the thesis of the theorem. So we can study cavity function to understand properties of the free energy. To do this, derive now the cavity function using the standard integration over Gaussian noise:

$$\partial_t \Psi(t) = \frac{1}{2}\mathbf{E}\left[1 - \frac{1}{N}\sum_i \omega_t^2(\sigma_i) \right] = \frac{1}{2}(1 - \langle q_{12}\rangle_t). \tag{25}$$

Note, being $Z_{N,0} = Z_N$, that the cavity function itself is obtainable as an integral over t of the simplest two-replicas overlap weighted with the generalized Boltzmann state ω_t.

$$\Psi_N(t,\omega) - \Psi_N(0,\omega) = \Psi_N(t,\omega) = \frac{1}{2}\int_0^t dt'(1 - \langle q_{12}\rangle_{t'}). \tag{26}$$

Let us show how to expand $q_{12} = \longleftrightarrow$: As the parameter of expansion we define the number of lines of the filled graphs; surely for the non-filled ones their parameter will be the corresponding one of the same order filled graph.

Before starting the expansion we present just some simple example:

es.1) $\langle \overset{\circ}{\bigcirc} \rangle$ this graph is of the fourth order, filled and means the quantity $\langle q_{12}^2 q_{34}^2 \rangle$.

es.2) $\langle \overset{\triangle}{\longleftrightarrow} \rangle$ this graph is of the fifth order, fillable and means the quantity $\langle q_{12}q_{23}q_{13}q_{45}\rangle$ (moreover its filled corresponding is $\langle \bigcirc\!\lhd \rangle$).

Remembering the streaming equation and the pull of rules developed before we can write:

$$\partial_t \langle \longleftrightarrow \rangle_t = \langle \bigcirc - 4 \longleftrightarrow\!\!\longleftrightarrow + 3 \overset{\longleftrightarrow}{\longleftrightarrow} \rangle_t \tag{27}$$

At this first step we find a filled graph, q_{12}^2; For it the perturbative expansion is ended here. Now we have to express also the other graphs (the non filled ones) in terms of the filled:

$$\partial_t \langle \longleftrightarrow \rangle_t = \langle \triangle + 2\bigcirc\!\!\rightarrow - 6\sqsubset - 3\curlywedge + 6\rightleftarrows \rangle_t \tag{28}$$

$$\partial_t \langle \leftrightarrows \rangle_t = \left\langle 2\ominus + 4\sqsubset - 16\rightleftarrows + 10\equiv \right\rangle_t . \tag{29}$$

Another filled one has been obtained (the first of the r.h.s. of the Eq. (28)). However we have to go on for the others.

$$\partial_t \langle \bigcirc\!\!\rightarrow \rangle_t = \left\langle \ominus\!\!\rightarrow + \bigcirc\!\!\curvearrowright + \bigcirc\!\bigcirc - 3\!\rightarrow\!\bigcirc\!\!\rightarrow - 3\bigcirc\!\!< - 3\bigcirc\!\!\rightarrow + 6\bigcirc\!\!\sqsubset \right\rangle_t \tag{30}$$

and so on. We stop here our expansion for now, showing just the result for higher orders. Neglecting terms order β^{10} we have:

$$\langle \longleftrightarrow \rangle_t = \left\langle \bigcirc t - 2\triangle t^2 - \frac{4}{3}\bigcirc\!\bigcirc t^3 + 8 t^3 + 6\square t^3 - +10\bigcirc\!\!\lhd t^4 - \frac{20}{3}\pentagon t^4 - 8\bigcirc\!\!\lhd t^4 \right\rangle . \tag{31}$$

We can now write (formally at any level of knowledge) the $\Psi(\beta)$ and, being known also the expression for the internal energy, we are able to write the expression for the free energy.

Proposition 4. *The free energy expansion via irreducible overlaps fluctuations is*

$$\alpha(\beta) = \Big\langle \ln 2 + \frac{\beta^2}{4}[1 + (1-\beta^2)\bigcirc] + \frac{\beta^6}{3}\triangle + \frac{\beta^8}{6}\bigcirc\!\bigcirc - \frac{\beta^8}{8} 8 - \frac{3\beta^8}{4}\square - \beta^{10}\bigcirc\!\!\lhd + \frac{12\beta^{10}}{5}\pentagon + \frac{2\beta^{10}}{3}\bigcirc\!\lhd \Big\rangle + O(\beta^{12}). \tag{32}$$

As we can immediately see this solution is built by more and more overlaps correlation functions (complete graphs) which a replica-symmetric theory cannot generate; moreover it is also easy to see that in the high temperature phase it reproduces the right expression and that the birth of all these correlations functions is just due to the entropy, being the energy density:

$$-\partial_\beta \alpha(\beta) = -\frac{\beta}{2}(1 - \langle \bigcirc \rangle) \tag{33}$$

while the entropy is:

$$S(\beta) = \left\langle \ln 2 - \frac{\beta^2}{4}(1 - 3\bigcirc) - \frac{\beta^4}{4}\bigcirc + \frac{\beta^6}{3}\triangle + \frac{\beta^8}{6}\bigcirc\!\bigcirc - \frac{\beta^8}{8} 8 \right\rangle + \cdots \tag{34}$$

4. OVERLAP CONSTRAINT GENERATOR

Now we want to use a pure thermodynamical approach to obtain the Aizenman-Contucci polynomials: we simply impose on the theory that the total derivative of the free energy with respect to β^2 has to be (in modulus) the internal energy of the system: (doing this we use the redundant formalism of $\alpha(\beta, \langle . \rangle_{\beta^2})$ instead of $\alpha(\beta)$ to put in evidence how to perform the right derivative.)

$$\alpha(\beta, \langle . \rangle_{\beta^2}) = \left[\ln 2 + \Psi(t, \langle . \rangle_t) - \frac{t}{4}(1 - \langle q_{12}^2 \rangle) \right]_{t=\beta^2}. \tag{35}$$

Its total derivative respect to β^2 is:

$$\frac{d}{d\beta^2}\alpha(\beta, \langle . \rangle_{\beta^2}) = \partial_{\beta^2}\alpha(\beta, \langle . \rangle_{\beta^2}) + \sum_{\langle . \rangle} \frac{\partial \alpha(\beta, \langle . \rangle_{\beta^2})}{\partial \langle . \rangle} \frac{\partial \langle . \rangle}{\partial \beta^2}, \tag{36}$$

where the expression $\sum_{\langle . \rangle}$ means *the sum over all the filled graphs contained in the* α and this derivative is equal (in modulus) to the internal energy:

$$\frac{d}{d\beta^2}\alpha(\beta, \langle . \rangle_{\beta^2}) = \frac{1}{2\beta}\alpha'_\beta = \frac{1}{4}\left(1 - \langle q_{12}^2 \rangle\right). \tag{37}$$

Start to calculate the total derivative: the partial one is

$$\partial_{\beta^2}\alpha(\beta, \langle . \rangle_{\beta^2}) = \left[\partial_t \Psi(t, \langle . \rangle) - \frac{1}{4}\left(1 - \langle q_{12}^2 \rangle\right) \right]_{t=\beta^2} \tag{38}$$

where

$$\partial_t \Psi(t, \langle . \rangle)_{t=\beta^2} = \lim_{t \to \beta^2} \frac{1}{2}\left(1 - \langle q_{12} \rangle_t\right) = \frac{1}{2}\left(1 - \langle q_{12}^2 \rangle\right). \tag{39}$$

So the partial and the total derivative are the same

$$\frac{d}{d\beta^2}\alpha(\beta, \langle . \rangle_{\beta^2}) = \partial_{\beta^2}\alpha(\beta, \langle . \rangle_{\beta^2}) \tag{40}$$

therefore the following quantity has to be identically zero:

$$\sum_{\langle . \rangle} \frac{\partial \alpha(\beta, \langle . \rangle_{\beta^2})}{\partial \langle . \rangle} \frac{\partial \langle . \rangle}{\partial \beta^2} = 0. \tag{41}$$

Giving an explicit expression we have:

Proposition 5. *The following expansion is the "thermodynamic" generator of the Aizenman–Contucci restrictions for the overlaps fluctuations:*

$$\left\langle \frac{\partial \alpha}{\partial \bigcirc}\frac{\partial \bigcirc}{\partial \beta^2} + \frac{\partial \alpha}{\partial \triangle}\frac{\partial \triangle}{\partial \beta^2} + \frac{\partial \alpha}{\partial \bigcirc\!\bigcirc}\frac{\partial \bigcirc\!\bigcirc}{\partial \beta^2} + \frac{\partial \alpha}{\partial \substack{\circ\\\circ\\\circ}}\frac{\partial \substack{\circ\\\circ\\\circ}}{\partial \beta^2} + \frac{\partial \alpha}{\partial \square}\frac{\partial \square}{\partial \beta^2} + \cdots \right\rangle = 0. \tag{42}$$

We try now to extrapolate some information from the above statement, starting to note that the apparently redundant word *thermodynamics* has been used to put in evidence the difference with the next overlap constraint generator which will be called *symmetric* because it will be derived just using the properties of the symmetry of the model.

Starting to study every single monomial we find a common structure to all members of the polynomial. For the first one:

$$\frac{\partial\alpha}{\partial\bigcirc}\frac{\partial\bigcirc}{\partial\beta^2} = \frac{\beta^2}{4}(1-\beta^2)\frac{1}{2\beta}\partial_\beta \mathbf{E}\Omega_s\left(q_{12}^2\right) \tag{43}$$

$$= Nf(\beta)\left[\left\langle\left(q_{12}^2\right)\left\{\left(\sum_{\alpha,\beta} q_{\alpha,\beta}^2 - s\sum_{\alpha} q_{\alpha,s+1}^2 + \frac{s(s+1)}{2}q_{s+1,s+2}^2\right)\right\}\right\rangle\right]. \tag{44}$$

For the second one:

$$\frac{\partial\alpha}{\partial\triangle}\frac{\partial\triangle}{\partial\beta^2} = \frac{\beta^6}{3}\frac{1}{2\beta}\partial_\beta \mathbf{E}\Omega_s(q_{12}q_{13}q_{23}) \tag{45}$$

$$= Ng(\beta)\left[\left\langle(q_{12}q_{13}q_{23})\left\{\left(\sum_{\alpha,\beta} q_{\alpha,\beta}^2 - s\sum_{\alpha} q_{\alpha,s+1}^2 + \frac{s(s+1)}{2}q_{s+1,s+2}^2\right)\right\}\right\rangle\right]. \tag{46}$$

We are now able to write the (42) in this other way:

$$\left\langle\left[\frac{\beta^2}{4}(1-\beta^2)\bigcirc + \frac{\beta^6}{3}\triangle + \frac{\beta^8}{6}\bigcirc\!\!\bigcirc + \frac{\beta^8}{8}\genfrac{}{}{0pt}{}{\circ}{\circ} - \frac{3\beta^8}{4}\square + \cdots\right]\times \times\left[\sum_{\alpha,\beta} q_{\alpha,\beta}^2 - s\sum_{\alpha} q_{\alpha,s+1}^2 + \frac{s(s+1)}{2}q_{s+1,s+2}^2\right]\right\rangle = 0. \tag{47}$$

We started taking as the parameter of the expansion the number of links inside a graph but, because there is a complete equivalence with the order of β^2, now we do the opposite: we think as the parameter β^2 and being the equation (42) a polynomial in β^2 equal to zero we have the right to put to zero every monomial inside. The first order give us:

$$\left\langle\frac{\beta^2}{4}(1-\beta^2)\bigcirc\left[\sum_{\alpha,\beta} q_{\alpha,\beta}^2 - s\sum_{\alpha} q_{\alpha,s+1}^2 + \frac{s(s+1)}{2}q_{s+1,s+2}^2\right]\right\rangle = 0 \tag{48}$$

If we put out from the averages the temperature function in order to neglect it when it's a well defined quantity we obtain:

$$\langle \ominus - 4\bigcirc\!\!\bigcirc + 3\,8 \rangle = 0 \tag{49}$$

This is the first Aizenman–Contucci relation or the first linear part of the Ghirlanda–Guerra identities. For the second order it's the same:

$$\langle \ominus\!\!\frown - 3\bigcirc\!\!\lhd + 2\bigcirc\!\lhd \rangle = 0 \tag{50}$$

Now we can start to look at the third order: we want to prove that the restrictions imposed by thermodynamic arguments are "sweet" with respect to the ones showed using symmetry properties of SK model Look at the expansion in β^2 of (47), thermodynamic requires that:

$$\left\langle \left[\frac{1}{6}\bigcirc\!\!\bigcirc - \frac{1}{8}8 - \frac{3}{4}\square \right] \left[\sum_{\alpha,\beta} q^2_{\alpha,\beta} - s\sum_{\alpha} q^2_{\alpha,s+1} + \frac{s(s+1)}{2} q^2_{s+1,s+2} \right] \right\rangle = 0. \tag{51}$$

Anyway, reading again the pull of rules developed before, we could state that:

Proposition 6. *The following expansion is the "symmetric" generator of the Aizenman–Contucci restrictions for the overlaps fluctuations:*

$$\lim_{N\to\infty} \lim_{t\to\beta^2} \partial_t[\text{filled graph}] = 0 \tag{52}$$

This property has been demonstrated implicitly because in the expansion previously showed we did not develop the graphs $q^2_{12}, q_{12}q_{23}q_{31}$ being them independent from the t parameter. Now we can look at this property also like a generator of constraints for the overlaps. Here we have some examples: Start with the first filled graphs and find again the linear Ghirlanda–Guerra relations:

$$\lim_{N\to\infty} \lim_{t\to\beta^2} \partial_t[q^2_{12}] = \left\langle \ominus - 4\bigcirc\!\!\bigcirc + 3\,8 \right\rangle = 0 \tag{53}$$

$$\lim_{N\to\infty} \lim_{t\to\beta^2} \partial_t[q_{12}q_{23}q_{31}] = \left\langle \ominus\!\!\frown - 3\bigcirc\!\!\lhd + 2\bigcirc\!\lhd \right\rangle = 0 \tag{54}$$

We can show how the derivative and the t-limit act together (in the thermodynamic limit) on a filled graph:

$$\lim_{N\to\infty} \lim_{t\to\beta^2} \partial_t \langle [\textit{filled graph}] \rangle = \left\langle [\textit{filled graph}] \left[\sum_{\alpha,\beta} q^2_{\alpha,\beta} - s\sum_{\alpha} q^2_{\alpha,s+1} + \frac{s(s+1)}{2} q^2_{s+1,s+2} \right] \right\rangle. \tag{55}$$

It's interesting that such constrictions, that are properties of SK model, are not strictly imposed by thermodynamic and are stronger that that ones: If we came back to look again at the fourth order (β^8) in our expansion we have that thermodynamic requires for the sum of the three filled graphs in the expression (51) (multiplied by the second parenthesis of the expression) to be zero but we have already show that also every single term in the above expression have to be zero. So the thermodynamic conditions are (respected and) weaker than that required by the symmetry of the model and this has the consequence that the system is not allowed to exploit the whole space but only the part of it where these restrictions hold. Moreover we found in this way a manner to build explicitly a kind of *constriction tree* for the overlap fluctuations because to obtain the restrictions at the fourth order we had to derive the unique graph of the second order and, after the limit of t that goes to β^2, we had to put that expression to zero obtaining linear GG identities. So to obtain the same constriction at the order six we can derive three of the four graphs of the fourth order (one is linearly dependent by the relation itself) and so on.

5. CONCLUSION AND OUTLOOK

In this paper we have used the method of the cavity fields via an interpolating parameter: we found this useful to derive an explicit expression for the free energy immediately below the critical temperature in terms of irreducible correlations functions of overlaps fluctuations (filled graphs). At the same time we obtained a way to generate systematically the constrictions to their free fluctuations (linear GG,[5,11]) and we found that, in the broken replica phase, the SK behavior is restricted mainly by its internal symmetries more than by thermodynamic stability. Future development should be finalized to apply this approach also to other mean field models with Gaussian quenched disorder and at the same time it should be useful to understand deeply which is the behavior of the tree of the constrictions at every non trivial order.

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